Continuity

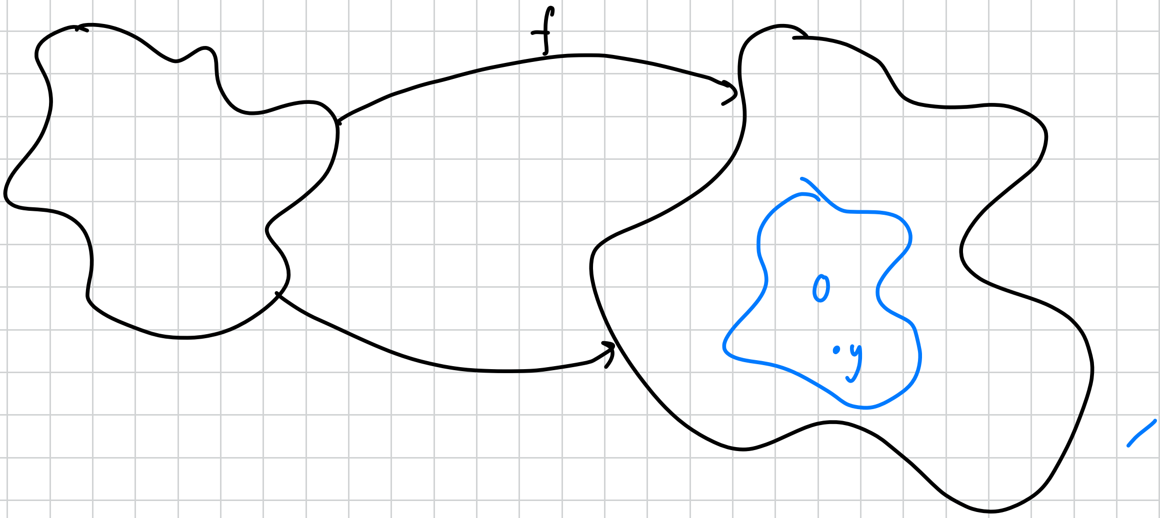
Preimages

Inverse image of a set under action of a function

Consider $f: X \to Y$

Let $A \subseteq Y$

$$f^{-1}(A) = \{x \in X : f(x) \in A\}$$



Fibre

Fibre for y

$$F_y = \{x \in X . f(x) = y\} \text{ , } y \in A \leftarrow F_y \neq \phi$$

Note: possible that no exists with $f(x) = y$

Suppose $F_y \neq \phi$. There is atleast one $x \in X$ s.t $f(x) = y$

$$\boxed{f^{-1}(A) = \bigcup_{y \in A} F_y}$$

Note:

- $f^{-1}(A)$ does NOT mean an inverse of f exists
- $f^{-1}(A)$ is the set of points in X which are mapped into A.

Continuity

Recall definition of f being cts in $\mathbb{R}$

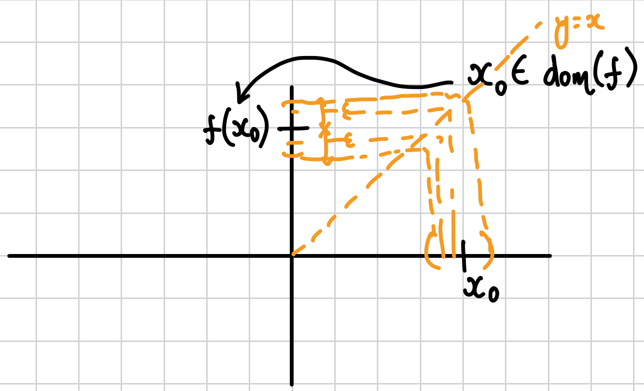
$$f:\mathbb{R}\to\mathbb{R}$$

Continuity in $\mathbb{R}$

For any $\varepsilon>0$, $\exists$ a $\delta>0$ s.t

$$|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon$$

Rough interpretation is that if you stay close to a point $x_0\in\mathbb{R}$, then we stay close to $f(x_0)$ in $\mathbb{R}$



We are going to define two notions of continuity

local continuity (cty at point $x_0\in X$)

global continuity (cty at all points in X)

LOCAL CONTINUITY IN METRIC SPACES

Definition of local continuity

Definition Local Continuity

Let (X, d_X) and (Y, d_Y) be metric spaces

Then

$f: X \to Y$ is continuous at $x_0 \in X \iff \forall \varepsilon > 0, \exists \delta > 0$ s.t

$d_Y(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$

Many faces of continuity

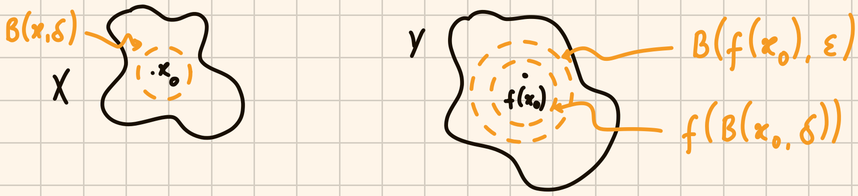
Theorem

Following statements are equivalent

(i) ε-δ defn given above

(ii) ε-δ ball version continuity

Given $\varepsilon > 0, \exists \delta > 0$ s.t $f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon)$

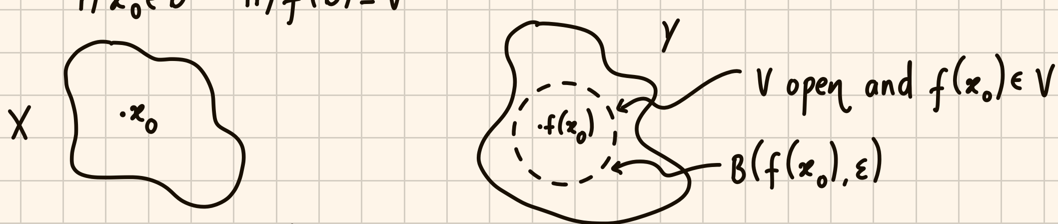


(iii) Open set defn of continuity at a point at x_0

Let V be open in Y and $f(x_0) \in V$.

Then $\exists$ open ball $B \subseteq X$ s.t

i) $x_0 \in B$ ii) $f(B) \subseteq V$



(iv) Let V open in Y and $f(x_0) \in V$. Then $\exists$ open set $U \subseteq X$ s.t

$x_0 \in U$ and $f(U) \subseteq V$

(v) For any sequences $(x_n)_{n=1}^{\infty}$ with $x_n \to x_0$

$f(x_n) \to f(x_0)$ as $n \to \infty$

Proof:

(iv) $\Rightarrow$ (v):

Let $x_0 \in X$ and $(x_n)_{n=1}^{\infty}$ a sequence in X such that $x_n \to x$ as $n \to \infty$

Assuming (iv) is true

Want to show that $f(x_n) \to f(x_0)$ as $n \to \infty$

Let V be open in Y and $f(x_0) \in V$. By iv) $\exists$ open set $U \subseteq X$ such that $x_0 \in U$ and $f(U) \subseteq$

U is open $\Longrightarrow \exists\, \varepsilon > 0$ such that $B(x_0, \varepsilon) \subseteq U$ (defn of open)

As $x_n \to x_0$, we can find $N > 0$ s.t $d(x_n, x_0) < \varepsilon \quad \forall\, n > N$ (defn of convergence)

Thus $f(x_n) \in f(U) \subseteq V \quad \forall\, n > N$

As V is arbitrary, $f(x_n) \to f(x_0)$ as $n \to \infty$

(v) $\Rightarrow$ (i): we will show the contrapositive

$$\sim(i) \Rightarrow \sim(v)$$

$(\sim i)$ $\exists\, \varepsilon > 0 \ \forall\, \delta > 0$ s.t $\hat{d}(f(x), f(x_0)) \geq \varepsilon$ for some $x \in X$ that satisfies

$$\delta(x, x_0) < \delta \qquad (\sim(i))$$

For each $n \in \mathbb{N}$, define the set

$$A_n = \{x' \in X : d(x', x_0) < 1/n \quad \& \quad \hat{d}(f(x'), f(x_0)) \geq \varepsilon\}$$

By the above, this set is non-empty. So choose an element from A_n and call this chosen element x_n.

Then $x_n \to x_0$ as $n \to \infty$ (as $d(x, x_0) < 1/n$) but $\hat{d}(f(x_n), f(x_0)) \geq \varepsilon > 0$

And so $f_n(x_n) \not\to f(x_0)$ (this establishes $\sim(v)$) ■

Useful property of preimages

Lemma:

Let $f: X \to Y$ be an arbitrary function and let $A \subseteq Y$ and $B \subseteq Y$. Then

$$f(A) \subseteq B \iff A \subseteq f^{-1}(B)$$

GLOBAL CONTINUITY IN METRIC SPACES

Global Continuity

Throughout $f: X \to Y$ where (X,d) and $(Y,\hat{d})$ are metric spaces

Definition Global continuity

$f: X \to Y$ is *globally* continuous if and only if it is locally continuous at every point $x_0 \in X$

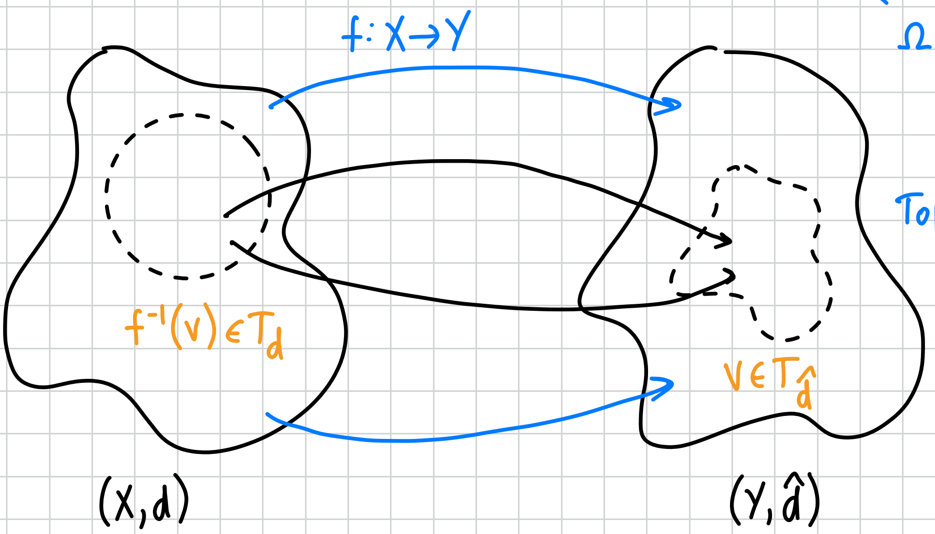
Definition: Topologists view of global continuity

Let $V \subseteq Y$ be open. Then, $\forall$ open sets $V \subseteq Y$,

f is *globally continuous* $\implies$ $f^{-1}(V) \subseteq X$ is open

Ω is open $\iff$ $\Omega = \emptyset$ or $\forall \Omega, \exists$ open ball $x \in B \subseteq \Omega$

Topology of (X,d) $T_d = \{\Omega \subseteq X \mid \Omega \text{ is open}\}$



An immediate equivalence from this defn is

$$\boxed{\begin{aligned} f \text{ cts } \forall x \in X &\iff f^{-1}(V) \in T_{\hat{d}} \text{ whenever } V \in T_d \\ &\iff \text{for any closed subset of } Y \text{, say } F \subseteq Y,\ f^{-1}(F) \text{ is closed} \end{aligned}}$$

Example: Application of definition:

Consider $f: \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto x^2 + e^y$, f is globally continuous

$\mathbb{R}^2$ equipped with d_2 metric

Consider

$$\Gamma = \{(x,y) \in \mathbb{R}^2 : f(x,y) = 1\} \quad \text{(preimage definition)}$$

$x^2 + e^y = 1$

Consider $\{1\} \subseteq \mathbb{R}$.

All singletons are closed and $f^{-1}(\{1\}) = \Gamma \implies \Gamma$ is closed by global continuity

Constant functions are continuous

Theorem Constant functions are continuous

Let $f: X \to Y: x \mapsto K$, K is fixed

Then f is continuous

Proof:

$Y = \{K\}$ and all singletons are closed.

$f^{-1}(\{K\}) = X$ and entire space is clopen $\implies X$ is closed

$\implies f$ is continuous ■

Composition of continuous functions are continuous

Theorem Composition of continuous functions are continuous

Suppose $f: X \to Y$ and $g: Y \to Z$ be continuous. Then

$$g \circ f : X \to Z$$

is continuous

Proof: Let $V \subseteq Z$ be open.

Then $g^{-1}(V) \subseteq Y$ is open $\Rightarrow f^{-1}(g^{-1}(V))$ is open in X (by continuity)

As $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$

It follows that $g \circ f$ is cts. ■

Note A composition can be continuous but its constituents may not

For example

$$f(t) = \begin{cases} 1 & t \geq 0 \\ -1 & t < 0 \end{cases}$$

$$g(t) = 0 \qquad \forall t \in \mathbb{R}$$

f is not continuous.

$$(g \circ f) = 0 \qquad \forall t \in \mathbb{R}$$

is continuous

Proving open set of global continuity

Theorem Global Continuity

$f: X \to Y$ is globally continuous if and only if

$\forall$ open sets $V \subseteq Y$, $f^{-1}(V) \subseteq X$ is open

Proof:

($\Longrightarrow$): (Using fibres)

Suppose that $f: X \to Y$ is globally continuous and consider an arbitrary open set $V \subseteq Y$.

CASE 1: $V \neq \emptyset$ then $f^{-1}(V) = \emptyset$ and $\emptyset$ is open

CASE 2: Suppose $V \neq \emptyset$ and let $y \in V$.

1) If $y \notin f(X)$, then $F_y = \emptyset$ and we are done

2) If $y \in f(X)$, then $F_y \neq \emptyset \Longrightarrow \exists x \in X$ such that $f(x) = y$

Since V is open and f is cts $\Longrightarrow \exists$ open ball $B(x, \varepsilon(x)) \subseteq X$ and $x \in X$ such that

$$f(B) \subseteq V$$

Therefore

$$f^{-1}(V) = \bigcup_{\substack{y \in V \\ F_y \neq \emptyset}} F_y = \bigcup_{x \in f^{-1}(V)} B(x, \varepsilon(x))$$

Since union of open balls are open, $f^{-1}(V)$ is open

($\Longleftarrow$): (Using epsilon-deltas)

Conversely suppose that $f^{-1}(V) \subseteq X$ is open $\forall$ open sets $V \subseteq Y \Longrightarrow \forall f(x) \in V$, $B(f(x), \varepsilon) \subseteq V$

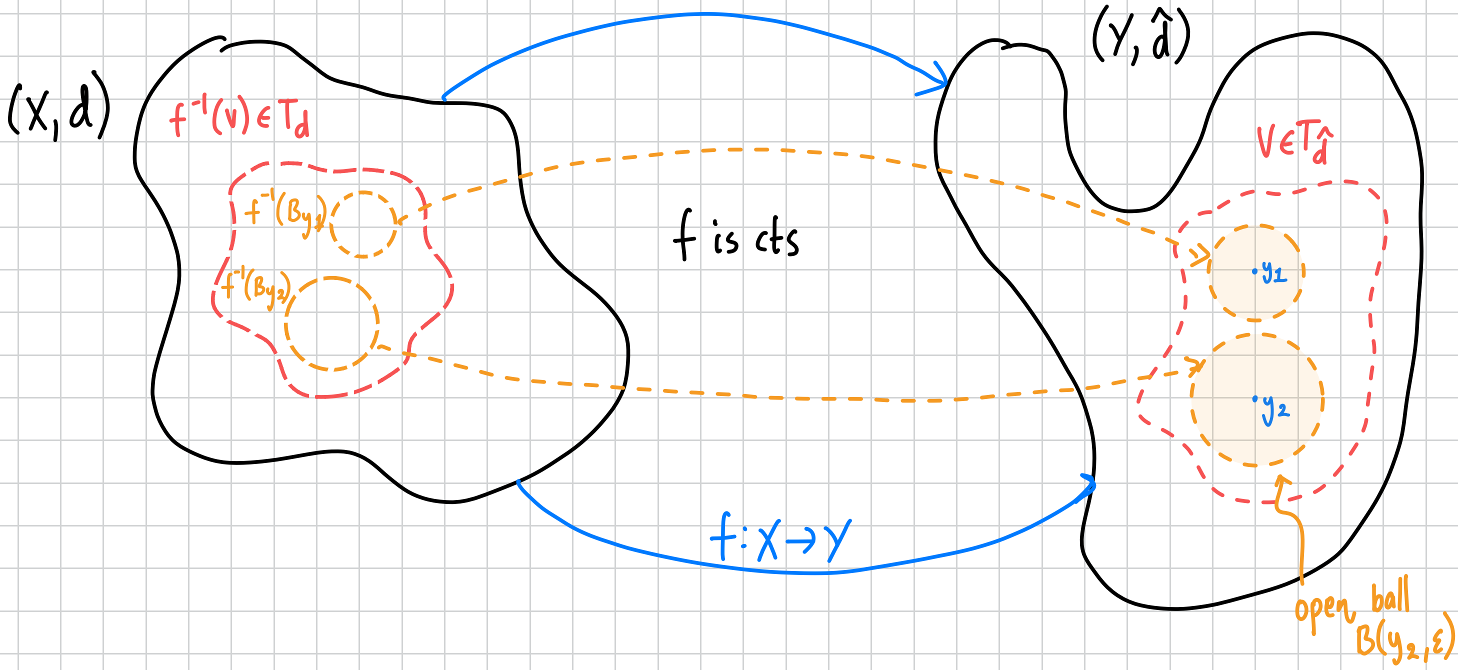
Take an $x \in X$

Note $B(f(x), \varepsilon)$ is open in $Y \Longrightarrow f^{-1}(B(f(x), \varepsilon))$ is open in X

Since $x \in f^{-1}(B(f(x), \varepsilon))$, $\exists \delta(\varepsilon) > 0$ such that $B(x, \delta) \subseteq f^{-1}(B(f(x), \varepsilon))$ (defn of open)

$$\Longrightarrow f \text{ is continuous at } x$$

Since x was arbitrary, f is globally continuous $\blacksquare$



Theorem: Global Continuity using closed sets

$f: X \to Y$ is globally continuous if and only if

$\forall$ closed sets $F \subseteq X$, $f^{-1}(F) \subseteq X$ is closed in X

Proof:

($\Rightarrow$): Suppose $F \subseteq Y$ is closed $\Longrightarrow F^c$ is open

$\Longrightarrow f^{-1}(F^c)$ is open continuity

$\Rightarrow f^{-1}(F)$ is closed look at set theory reference

($\Leftarrow$): Suppose V is open $\Longrightarrow V^c$ closed

$\Longrightarrow f^{-1}(V^c)$ closed

$\Rightarrow f^{-1}(V)$ open

$\Rightarrow$ continuous ■

Example: Recall from the section on convergence, if we have k sequences of real numbers, $(x_n^{(i)})$ s.t $x_n^{(i)} \to x_i$ as $n \to \infty$. Then

$\underline{x}_n \in \mathbb{R}^k$ (equiped with d_∞)

where $\underline{x}_n = (x_n^{(1)}, x_n^{(2)}, x_n^{(3)}, \dots, x_n^{(k)})$ then

$\underline{x}_n \to \underline{x} = (x_1, \dots, x_n)$

Note: f is globally continuous $\iff$ f is cts at all $x \in X$

$\iff$ for all $x \in X$, if $x_n \to x$ as $n \to \infty$, then $f(x_n) \to f(x)$ as $n \to \infty$

Theorem

Let $f_1, f_2, \dots, f_K$ be continuous functions from (X,d) to $(\mathbb{R}^K, d_\infty)$

Then
$$F: X \to \mathbb{R}^K \,;\, x \mapsto (f_1(x), f_2(x), \dots, f_K(x)) \text{ is continuous}$$

Proof: Suppose x_n is a sequence in X and
$$x_n \to x \quad \text{as } n \to \infty$$

As f_i is continuous, $f_i(x_n) \to f_i(x)$ (sequential characterization of continuity)

$$\implies F(x_n) \to F(x) \quad \text{where } F(x) = (f_1(x), \dots, f_n(x))$$

(component wise convergence $\implies$ total convergence in d_∞)

$\blacksquare$

BOUNDED FUNCTIONS AND UNIFORM CONVERGENCE

Discuss: Consider

$$C(X,Y) = \{ f : X \to Y : f \text{ is cts}\}$$

Try to generalise $C([0,1],\mathbb{R})$ and $\underbrace{d_1, d_2}$, d_∞ ← possible !!!

↳ not possible ! integration may not exist / defined

Take $C([0,1])$ and d_∞

$$d_\infty(f,g) = \sup\{ |\underbrace{f(x) - g(x)}| : x \in [0,1]\}$$

replace with $\hat{d}(f(x), g(x))$

$\hat{d}$ is a metric on Y

Attempt $C(X,Y)$

(X,d) $(Y,\hat{d})$

$$d(f,g) = \sup\{ \hat{d}(f(x), g(x)) : x \in X\}$$

Problem: $d_\infty(f,g) \notin [0,\infty)$ (may or may not; force bounded)

Definition: Bounded metric space

A metric space (X,d) is **bounded** if and only if

$\exists\, M \in \mathbb{R}$ such that $d(a,b) \leq M \quad \forall\, a,b \in X$

Bounded functions

Bounded: $f : \mathbb{R} \to \mathbb{R}$ is bounded $\Leftrightarrow \exists\, M > 0$ s.t $|f(x)| < M \quad \forall\, x \in \mathbb{R}$

has metric interpretation

Definition: Bounded functions

Let $f: X \to Y$ where X and Y are metric spaces.

Then f is bounded $\iff \exists$ open ball $B \subseteq Y$ s.t $f(x) \in B \; \forall x \in X$

$\iff \exists z \in Y$ and $M \in (0, \infty)$ s.t $f(X) \subseteq B(z, M)$

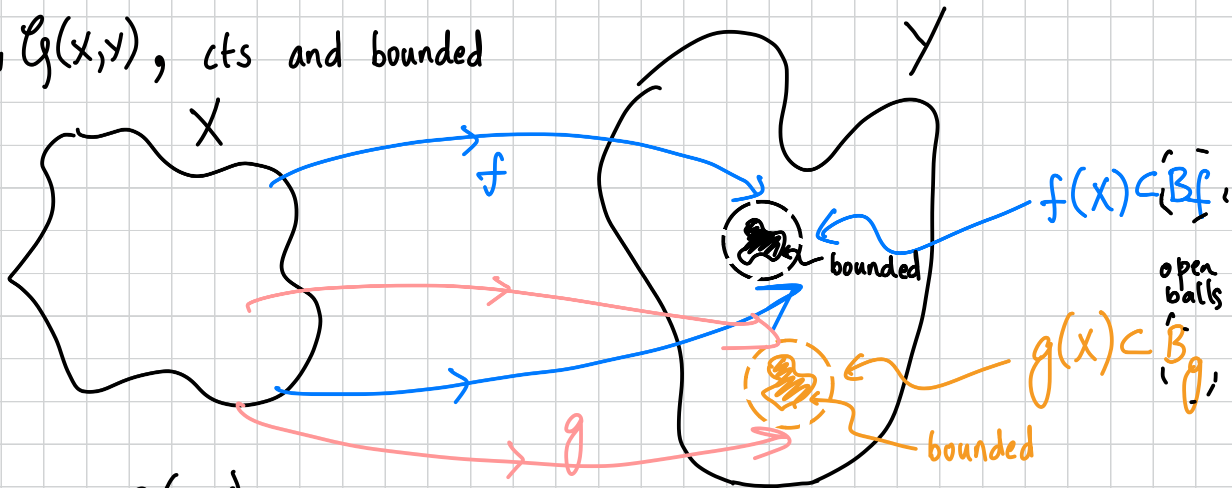
Let $\mathcal{C}(X,Y)$ be the set of all bounded AND continuous functions

$$f: X \to Y$$

i.e.

$$\mathcal{C}(X,Y) = C(X,Y) \cap B(X,Y)$$

Working in $\mathcal{C}(X,Y)$, cts and bounded



Consider a space $B(X,Y)$ all bounded functions $f: X \to Y$

The uniform (sup) metric on $B(X,Y)$ is

$$d_\infty(f,g) = \sup\{\hat{d}(f(x), g(x)) \mid x \in X\}$$

is a metric on the sub-space $\mathcal{C}(X,Y)$

$c(X,Y)$ the set of all cts functions from (X,d) to $(Y,\hat{d})$

$\boxed{B(X,Y) \text{ is the set of all bounded functions from } X \to Y}$

Definition Open ball defn of bounded

A function is bounded $\iff$ $\exists$ open ball $B \subseteq Y$ such that $f(X) \subseteq B$

This means that $\exists z \in Y$ and $R \in (0,\infty)$ s.t $f(X) \subseteq B(z,R)$

↳ as a consequence, if

$\underset{\in X}{x, x'}$ and $\underset{\in Y}{f(x), f(x')}$

then

$$\hat{d}(f(x), f(x')) \leq \hat{d}(f(x), z) + \hat{d}(f(x'), z) < 2R$$

We were able to generalise from d_∞ from $c([0,1], \mathbb{R}) \longrightarrow c([0,1] \to \mathbb{R})$

$$\boxed{\underset{\in B(X,Y)}{d_\infty(f,g)} = \sup\{\hat{d}(f(x), g(x)) \mid x \in X\}}$$

uniform or sup metric on $B(X,Y)$

Now define $\boxed{\mathcal{C}(X,Y) = B(X,Y) \cap c(X,Y)}$ ← continuous and bounded functions

$\implies (\mathcal{C}(X,Y), d_\infty)$ is a metric space

Consider $(\mathcal{C}(X, \mathbb{K}), d_\infty)$; $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

d_∞ induces a stronger notion of continuity and convergence.

Uniform Convergence

Definition Uniform Convergence

Let $(f_n)_{n=1}^{\infty}$ be a sequence from $(\mathcal{C}(X,\mathbb{K}), d_\infty)$

So $f_n \to f$ uniformly as $n\to\infty \Leftrightarrow \forall \varepsilon>0, \exists N>0$ s.t $d_\infty(f_n,f)<\varepsilon \; \forall n>N$

Note: as $d_\infty(f_n,f) = \sup\{\underbrace{|f_n(x)-f(x)|}_{\text{std metric on } \mathbb{K}} \; x\in X\}$

↳ This is independant of $x\in X$, as $|f_n(x)-f(x)|<\varepsilon \; \forall x\in X$

Contrast this with pointwise convergence of f_n.

Take $x\in X$ and fix it.

Consider sequence $(f_n(x))_{n=1}^{\infty} \in \mathbb{K}^{\mathbb{N}}$

$f_n \to f$ pointwise $\iff f_n(x)\to f(x)$ as $n\to\infty$ for any $x\in X$

Theorem

Uniform Convergence $\Rightarrow$ Pointwise Convergence

Note: Pointwise $\not\Rightarrow$ uniform. But if $f_n\to f$ pointwise and f is either not bounded or cts

$\Rightarrow f_n \not\to f$ is uniformly

Example: of pointwise $\not\Rightarrow$ uniform

Take $(f_n)_{n=1}^{\infty}$ $\mathcal{C}([0,1],\mathbb{R})$

where

$$f_n(t) = \begin{cases} 0 & \text{if} \quad t=0 \\ t^n & \text{if} \quad t<1 \end{cases}$$

$d_\infty(f_n,0)=1 \not\to 0$ as $n\to\infty$, $f_n(t)\to 0$ as $n\to\infty$ for any $t\in[0,1]$

Recall (X,d) $x_n\to x$ as $n\to\infty \iff d(x_n,x)\to 0$ as $n\to\infty$

Definition Uniformly convergent

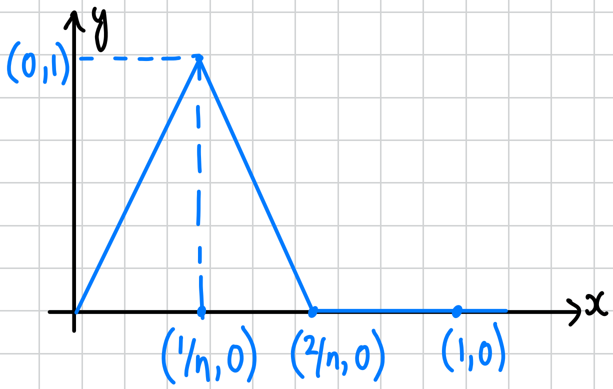
A sequence of functions $(f_n)_{n=1}^{\infty}$ is **uniformly convergent** if
$$d_\infty(f_n, f) \to 0 \quad \text{as } n \to \infty$$

Example 2: Pointwise $\not\Rightarrow$ uniform

Let $X = Y = \mathbb{R}$ and f_n be defined by
$$f_n(x) = \begin{cases} 0 & \text{if } x \le 0 \\ nx & \text{if } 0 \le x \le 1/n \\ -nx+2 & \text{if } 1/n \le x \le 2/n \\ 0 & \text{if } x \ge 2/n \end{cases}$$

and $f(x) = 0 \quad \forall x \in \mathbb{R}$.

Then $f_n \to f$ pointwise on $\mathbb{R}$



If x is close to 0, $\exists n_0 \in \mathbb{N}$ s.t $\forall n > n_0$
$$|nx - 0| < \frac{1}{2} \Rightarrow \frac{2}{n_0} < x$$
so
$$f_n(x) = 0 \quad \forall n \ge n_0$$

However no such n_0 exists such that
$$|f_n(x) - f(x)| < \frac{1}{2} \quad \forall n \ge n_0 \quad \forall x \in \mathbb{R}.$$

If this was true then for $0 \le x \le 1/n_0$
$$f_n(x) < \frac{1}{2} \Rightarrow n_0 x < 1/2$$

This must be true for any $x \in \mathbb{R}$, but if we have $x = \frac{2}{3n_0}$
$$\frac{2}{3n_0} < \frac{1}{2n_0} \Rightarrow \frac{2}{3} < \frac{1}{2}$$
$$\Rightarrow \text{contradiction}$$

Definition pointwise convergence

Let $f: X \to Y$ and $f_n: X \to Y$ be given, $n = 1, 2, \cdots$, be given.

We say that $\{f_n\}$ **converges pointwise** to f iff

$$\lim_{n\to\infty} d_Y(f_n(x), f(x)) = 0 \qquad \forall x \in X$$

for any **fixed** $x \in X$

The ε-δ definition is

$\{f_n\}_{n\geq 1} \to f$ pointwise $\iff$ for a given $\varepsilon > 0$, given $x \in X$, $\exists N = N(x, \varepsilon)$ such that

$$d_Y(f_n(x), f(x)) < \varepsilon \qquad \forall n \geq N$$

$N = N(x, \varepsilon)$ depends on x and ε.

Uniform convergence

Definition Uniform convergence

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of mappings from (X, d_X) to (Y, d_Y).

$$f_n: (X, d_X) \longrightarrow (Y, d_Y).$$

We say that the sequences $\{f_n\}$ **converges uniformly** on X to a mapping

$$f: X \to Y$$

if $\forall \varepsilon > 0$, $\exists N \in N(\varepsilon)$ such that

$$d_Y(f_n(x), f(x)) < \varepsilon$$

for all $n \geq N$ and all $x \in X$, i.e.

$$\lim_{n\to\infty} \left(\sup_{x\in X} d_Y(f_n(x), f(x)) \right) = 0$$

Alternatively we say

$\{f_n\}_{n=1}^{\infty} \to f$ uniformly if

$$d_\infty(f_n, f) \to 0 \quad \text{as } n \to \infty$$

$$d_\infty(f_n, f) = \sup_{x\in X} \{ d_Y(f_n(x), f(x)) \mid x \in X \}$$

Theorem:

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \to Y$.

Then the following are equivalent

(i) f is continuous on X

(ii) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all subsets B of Y

(iii) $f(\overline{A}) \subseteq \overline{f(A)}$ for all subsets A of X

Proof:

(i) $\Rightarrow$ (ii):

Consider any arbitrary $B \subseteq Y$.

$\overline{B}$ is a closed subset of $Y \Rightarrow f^{-1}(\overline{B}) \subseteq X$ is closed (defn of continuity)

Further $B \subseteq \overline{B}$ (defn of closure)

$$\Rightarrow f^{-1}(B) \subseteq f^{-1}(\overline{B})$$

and therefore

$$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$$

closure is the smallest superset that is closed

(ii) $\Rightarrow$ (iii)

Let $A \subseteq X$. Then

if $B = f(A) \Rightarrow A \subseteq f^{-1}(B)$ $\quad$ $A \subseteq f^{-1}(f(A))$

$\Rightarrow \overline{A} \subseteq \overline{f^{-1}(B)}$ (Closure property $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$)

$\Rightarrow \overline{A} \subseteq \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ $\quad$ by (ii)

$\Rightarrow \overline{A} \subseteq f^{-1}(\overline{B})$

$\Rightarrow f(\overline{A}) \subseteq f(f^{-1}(\overline{B}))$

$\Rightarrow f(\overline{A}) \subseteq \overline{B} = \overline{f(A)}$ $\quad$ $(B = f(A) \Rightarrow \overline{B} = \overline{f(A)})$

$\Rightarrow f(\overline{A}) \subseteq \overline{f(A)}$

(iii) $\Rightarrow$ (i)

Suppose that $F \subseteq Y$ and F is closed. Then

$$f^{-1}(F) = F_1$$

We need to show that F_1 is closed, i.e. $F_1 = \overline{F_1}$

By (iii),

$$f(\overline{F_1}) \subseteq \overline{f(F_1)} \Rightarrow f(\overline{F_1}) \subseteq \overline{f(f^{-1}(F))}$$ defn of F_1 defined above

$$\Rightarrow f(\overline{F_1}) \subseteq F = \overline{F}$$ (F is closed)

preimage of an image $f(f^{-1}(B)) = B$

Therefore

$$\overline{F_1} \subseteq f^{-1}(f(\overline{F_1})) \Rightarrow f^{-1}(f(\overline{F_1})) \subseteq f^{-1}(F) = F_1$$

since $A \subseteq f^{-1}(f(A))$

CONTRACTIONS AND CONTRACTION MAPPING THM

Motivation: Solving

$$f(x) = \alpha$$

Newtons root finding algorithm.
- start with a guess
- construct a sequene of better guess
- note that sequence converges.

Turn the problem into a fixed point problem.

$$f(x) = \alpha \iff f(x) - \alpha = 0$$
$$\iff f(x) + x - \alpha = x$$
$$\iff g(x) = x$$

(define $g(x) = f(x) - \alpha + x$)

We need a further property of f to define contractions.

Let $f:(X,d) \to (Y,\hat{d})$ be a metric space.

Definition Lipchitz

f is Lipchitz function $\iff \exists\, k\,[0,\infty)$ s.t $\hat{d}(f(x), f(x')) \leq k\, d(x,x')$

The constant k is called Lipchitz constant

Definition Contraction

A (strict) contraction is a Lipchitz function for which $k \in [0.1)$

Example:

$(\mathbb{R}^k, d_2)$ and $f: \mathbb{R}^k \to \mathbb{R}^k$

$$\underline{x} \mapsto \lambda \underline{x}$$

a contraction providing $|\lambda| < 1$.

Theorem Contraction mapping thm (Bernach fixed point thm)

Let (X,d) be a complete metric space, and $f: X \to X$ be a contraction.

Then f has a fixed point say $y \in X$

Take any $x_0 \in X$, the sequence $(x_n)_{n=1}^{\infty}$ where

$$\underset{\text{iterates}}{x_n} = f(x_{n-1}) \qquad x \geq 1$$

converges to y. That is

$$x_n \to y \text{ as } n \to \infty \text{ for any } x_0$$

Proof: Take any $x_0 \in X$, and define sequence

$$x_n = f(x_{n-1}) \qquad \forall n \geq 1$$

As f is a strict contraction, there exists $k \in [0,1)$ such that

$$d(f(x), f(x')) \leq k\, d(x, x') \qquad \text{for all } x, x' \in X$$

Aim: Show that $(x_n)_{n=1}^{\infty}$ is Cauchy in which case $\exists y \in X$ s.t $x_n \to y$ as $n \to \infty$ (completeness of X)

Consider

$$\begin{aligned} d(x_n, x_{n-1}) &= d(f(x_{n-1}), f(x_{n-2})) \\ &\leq k\, d(x_{n-1}, x_{n-2}) \qquad \text{(strict contraction)} \\ &\leq k.k\, d(x_{n-2}, x_{n-3}) \leq k^2 d(x_{n-2}, x_{n-3}) \\ &\;\;\vdots \\ &\leq k^{n-1} \underbrace{d(x_1, x_0)}_{\text{constant once } x_0 \text{ is fixed}} \end{aligned}$$

Consider $m > n$ ($m = n + l$ for some $l \in \mathbb{N}$)

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_n) \\ &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + d(x_{m-2}, x_n) \\ &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq k^{m-1} d(x_1, x_0) + k^{m-2}(x_1, x_0) + \cdots + k^n d(x_1, x_0) \\ &= d(x_1, x_0)\left(k^{m-1} + k^{m-2} + \cdots + k^n\right) \end{aligned}$$

repeated use of Δ-inequality

$$= k^n d(x_1,x_0)\left(1+k+\dots+k^{m-1-n}\right)$$
$$= k^n d(x_1,x_0)\left(1+k+\dots+k^{l-1}\right) \le k^n d(x_0,x_1)(1+k+\dots+k^{l-1}+k^l+\dots)$$

infinite

Recall geometric series with common ratio r is a series of the form
$$1+r+r^2+\dots+r^l \longrightarrow \frac{1}{1-r} \quad \text{if } |r|<1$$
$\rightarrow$ true as contraction $\Rightarrow k\in[0,1)$

Thus
$$d(x_m,x_n) \le k^n d(x_1,x_0)\left(1+k+k^2+\dots+k^l+\dots\right) \le \frac{k^n}{1-k} d(x_1,x_0)$$

So
$$d(x_m,x_n) \le k^n\left(\frac{d(x_1,x_0)}{k-1}\right)$$
$$= k^n\lambda \quad \text{where } \lambda = \frac{d(x_1,x_0)}{k-1} \text{ is a constant}$$
$$\to 0 \text{ as } n\to\infty \quad (\text{as } |k|<1)$$

Thus $(x_n)_{n=1}^{\infty}$ is Cauchy.

Completeness tells us that $\exists y\in X$ s.t $x_n\to y$ as $n\to\infty$.

Now we show y is a fixed point; $y=f(y)$

Recall contractions are cts. So
$$x_n\to y \text{ as } n\to\infty \Rightarrow f(x_n)\to f(y) \text{ as } n\to\infty$$
continuity

We know that $x_n\to y$. But
$$x_1=f(x_0),\ x_2=f(x_1),\ x_3=f(x_2),\ x_4=f(x_3),\ \dots,\ x_{n+1}=f(x_n)$$
$f(x_0), f(x_1), \dots, f(x_n)$ is just $x_1, x_2, x_3, \dots$

As limits are unique,
$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n$$

Given $\lim_{n\to\infty} f(x_n) = f(y)$ by continuity and $\lim_{n\to\infty} x_n = y \Rightarrow f(y)=y$

Uniqueness: Suppose that y' also a fixed point of f, i.e.
$$f(y')=y'$$

Consider
$$d(y,y') = d(f(y),f(y')) \le k\, d(y,y')$$

This only holds if $y = y'$ otherwise we'd get

$$d(y, y') > 0 \quad \text{and} \quad d(y, y') < d(y, y')$$

$\blacksquare$

Examples applying contraction mapping thm

1) Take $f : \mathbb{R} \to \mathbb{R}$

$$t \mapsto \frac{1}{2}\sqrt{t^2+1}$$

Show that f is a contraction and hence the sequence $x_0, x_1 = f(x_0), x_2 = f(x_1), \dots$ converges and find the limit

Notice that $f(\mathbb{R}) = [1/2, \infty)$ ← take any real t. Then $t^2 \geq 0 \Rightarrow t^2+1 \geq 1$

$$\Rightarrow \sqrt{t^2+1} \geq 1$$

$$\Rightarrow \frac{1}{2}\sqrt{t^2+1} \geq \frac{1}{2}$$

So we restrict f to the subspace $[1/2, \infty)$. Then

$$f : [1/2, \infty) \to [1/2, \infty)$$

Recall the Mean Value Thm

Theorem Mean Value Thm

Let $g : [a,b] \to \mathbb{R}$ be continuous and differentiable. Then for any $x<y$ in $[a,b]$, $\exists\, c \in (x,y)$ s.t

$$f'(c) = \frac{f(y)-f(x)}{y-x}$$

Method:

$$f'(c) = \frac{f(y)-f(x)}{y-x} \quad \Rightarrow \quad |f'(c)| = \frac{|f(y)-f(x)|}{|y-x|}$$

$$\Rightarrow \quad |y-x||f'(c)| = |f(y)-f(x)|$$

$$\Rightarrow \quad d(y,x)\underbrace{|f'(c)|}_{\text{want } <1} = d(f(y), f(x)) \quad \Rightarrow \quad \boxed{\begin{array}{c}\text{optimal lipchitz constant} \\ \max/\sup\limits_{c \in [a,b]} |f'(c)|\end{array}}$$

Consider $f'(t) = \frac{1}{2} \cdot 2t \frac{1}{\sqrt{t^2+1}}$

$$= \frac{1}{2}\sqrt{\frac{t^2}{t^2+1}} < \frac{1}{2} \qquad \forall\ t \in [1/2, \infty)$$

Take any $x<y$ in $[1/2, \infty)$. By MVT

$$|y-x| \frac{1}{2} \geq |f(y) - f(x)|$$

Thus f a contraction.

Thus far, we have shown $f : [1/2, \infty) \to [1/2, \infty)$:

$$t \mapsto \frac{1}{2}\sqrt{t^2+1}$$

is a contraction with Lipchitz constant $1/2$.

We know that $C([a,b])$ equipped with d_∞ metric is complete. So we can use CMT

$$\Longrightarrow \exists \text{ a unique fixed point } y \text{ of } f.$$

We know from the proof of contraction mapping thm that the sequences

$$x_0, x_1, x_2, \cdots \quad \text{and} \quad f(x_0), f(x_1), \ldots$$

both converge to y.

But by continuity of f, $f(y) = y$

So we need to solve $f(y) = y \Rightarrow y = \frac{1}{2}\sqrt{y^2+1} \Rightarrow 2y = \sqrt{y^2+1}$

$$\Rightarrow 4y^2 = y^2 + 1$$

$$\Rightarrow 3y^2 = 1$$

$$\Rightarrow y = \frac{1}{\sqrt{3}}$$

Hidden treat in proof of CMT

$$d(x_m, x_n) \leq k^{n-1}\left(\frac{d(x_1, x_0)}{k-1}\right) \xRightarrow{\text{let } m \to \infty} d(y, x_n) \leq k^{n-1}\left(\frac{d(x_1, x_0)}{k-1}\right)$$

2) Fredholm equations (found in signal and image processing)

(of the second kind)

$$\boxed{f(t) = \underbrace{v(t)}_{\text{given}} + \frac{1}{\lambda}\int_a^b \underbrace{k(t,s)}_{\text{given (kernal)}} f(s)\, ds}$$

<u>Assumptions</u>: • v is continuous on $[a,b]$

• k is continuous on $[a,b]^2$

<u>Note</u>: $$\int_a^b k(t,s) f(s)\, ds = F(t)$$

Can we turn this into a fixed point problem and use contraction mapping thm.

Work in $C([a,b],\mathbb{R})$ equipped with d_∞ metric

(Q) is this complete? Ans: Yes)

As k is continuous on $[a,b]^2$, then $\exists\, M>0$ such that

$$|k(t,s)| \le M \qquad \forall \quad t,s \in [a,b]$$ Extreme Value thm

Define the function T on $C([a,b])$ by

$$(Tf)(t) = v(t) + \underbrace{\int_a^b k(s,t) f(s)\, ds}_{F(t)}$$

So a solution to

$$f(t) = v(t) + \int_a^b k(t,s)\, f(s) = (Tf)(t)$$

So a solution f to our original Freedholm equation is a fixed point of T

$$T: C([a,b]) \to C([a,b])$$

Apply CMT!!!

$$|(Tf)(t)-(Tg)(t)| = \left|\cancel{v(t)}+\frac{1}{\lambda}\int_a^b K(t,s)f(s)\,ds - \cancel{v(t)} - \frac{1}{\lambda}\int_a^b K(t,s)g(s)\,ds\right| \quad (*)$$

$$= \left|\frac{1}{\lambda}\int_a^b K(t,s)(f(s)-g(s))\,ds\right|$$

$$\leq \frac{1}{|\lambda|}\int_a^b |K(t,s)|.|f(s)-g(s)|\,ds \quad \text{(triangle inequality for integrals)}$$

$$\leq \frac{1}{|\lambda|}\int_a^b M.|f(s)-g(s)|\,ds$$

$$= \frac{M}{|\lambda|}\int_a^b |f(s)-g(s)|\,ds$$

But $d_\infty(f,g) = \sup\{|f(s)-g(s)| : s\in[a,b]\}$ and $|f(s)-g(s)| \leq d_\infty(f,g) \quad \forall s\in[a,b]$

↳ supremum defn

and therefore

$$(*) \leq \frac{M}{|\lambda|}\int_a^b |f(s)-g(s)|\,ds \leq \frac{M}{|\lambda|}\int_a^b d_\infty(f,g)\,ds$$

$$= \frac{M}{|\lambda|}d_\infty(f,g)\int_a^b 1\,ds$$

$$= \frac{M}{|\lambda|}d_\infty(f,g)(b-a) \quad \text{(integrals min-max inequality)}$$

and we have

$$\sup\{|(Tf)(t)-(Tg)(t)|\ t\in[a,b]\} = d_\infty(Tf,Tg)$$

$$\leq \frac{M}{|\lambda|}d_\infty(f,g)(b-a)$$

$$= \frac{M(b-a)}{|\lambda|}d_\infty(f,g)$$

so if we choose λ to satisfy

$|\lambda| > M(b-a)$ then T is a contraction

Prove that $(C[(a,b)], d_\infty)$ is complete.

Proposition

The metric space $C([a,b])$ equipped with d_∞ metric

$$d_\infty(f,g) = \sup\{|f(t)-g(t)| \; t\in[a,b]\}$$

is complete

Proof:

Let $(f_n)_{n=1}^{\infty}$ be any arbitrary Cauchy sequence

Thus for any $\varepsilon>0$, $\exists\, N>0$, s.t $d_\infty(f_n,f_m)<\varepsilon \quad \forall\, n,m>N$

$$d_\infty(f_n,f_m)<\varepsilon \implies \sup\{|f_n(t)-f_m(t)| \; t\in[a,b]\}<\varepsilon$$

$$\implies |f_n(t)-f_m(t)|<\varepsilon \;\; \forall\, m,n>N \text{ and all } t\in[a,b]$$

$$\implies \{f_n(t)\}_{n\geq 1} \text{ is Cauchy for any fixed } t\in[a,b]$$

sequence of real numbers

But $\mathbb{R}$ is complete $\implies \{f_n(t)\}_{n\geq 1}$ converges.

$$\implies \exists\, f_t\in\mathbb{R} \text{ such that } f_n(t)\to f_t \text{ as } n\to\infty$$

Construct candidate limit

$$f:[a,b]\to\mathbb{R}\,; \quad f(t)=f_t$$

We need to show that

i) $f_n\to f$ as $n\to\infty$

ii) f is continuous $\implies f\in C[a,b]$

(i) Showing $f_n\to f$ as $n\to\infty$.

We need to show that for any given $\varepsilon>0$, $\exists\, N=N(\varepsilon)$ s.t

$$d_\infty(f_n,f)<\varepsilon \quad \forall\, n>N.$$

Since by assumption $\{f_m\}$ is Cauchy,

$$d_\infty(f_n,f_m)<\varepsilon \qquad \forall\, m,n>N.$$

$$d_\infty(f_n,f_m)<\varepsilon \implies \sup\{|f_n(t)-f_m(t)| : t\in[a,b]\}<\varepsilon$$

$$\Rightarrow \quad |f_n(t)-f_m(t)| < \varepsilon \quad \forall\, m,n > N \text{ and all } t \in [a,b]$$

Now consider $|f_n(t)-f(t)|$

$$\begin{aligned} |f_n(t)-f(t)| &= |f_n(t)-f_m(t)+f_m(t)-f(t)| \\ &\leq |f_n(t)-f_m(t)| + |f_m(t)-f(t)| \qquad \text{triangle inequality} \\ &< \varepsilon + |f_m(t)-f(t)| \end{aligned}$$

Since as shown above $\{f_m(t)\}_{m\geq 1}$ converges to $f(t)$ as $m\to\infty$ and therefore $|f_m(t)-f(t)| \to 0$ as $m\to\infty$ and hence

$$|f_n(t)-f(t)| \leqslant \varepsilon \quad \forall\, n > N, \text{ and all } t\in[a,b]$$

Hence we get

$$d_\infty(f_n, f) \leqslant \varepsilon \quad \forall\, n > N$$

$$\Rightarrow \quad f_n \to f \text{ as } n\to\infty$$

(ii) Showing that f is continuous, fix $t_0 \in [a,b]$

We need to show that

$$\lim_{t\to t_0} f(t) = f(t_0) \iff \forall\, \varepsilon>0,\ \exists\, \delta=\delta(\varepsilon) \text{ s.t } \forall\, |t-t_0|<\delta,\ |f(t)-f(t_0)|<\varepsilon$$

a) Since $f_n \to f$ uniformly (shown above), choose $N = N(\varepsilon)$ such that $\forall\, n > N_\varepsilon$,

$$d_\infty(f_n, f) < \varepsilon/3 \Rightarrow |f_n(t)-f(t)| < \varepsilon/3 \quad \forall\, t\in[a,b]$$

(b) Since f_n is continuous as $\{f_n\}_{n=1}^\infty$ is a sequence on $C[a,b]$,

$$|f_n(t)-f_n(t_0)| < \frac{\varepsilon}{3}$$

(c) And further, since $\{f_n(t_0)\}_{n\geq 1}$ is Cauchy, it converges by completeness of $\mathbb{R}$

$$\Rightarrow \quad |f_n(t_0)-f(t_0)| < \varepsilon$$

Therefore for if $|t-t_0|<\delta$,

$$\begin{aligned} |f(t)-f(t_0)| &= |f(t)-f_n(t)+f_n(t)-f_n(t_0)+f_n(t_0)-f(t_0)| \\ &\leq |f(t)-f_n(t)| + |f_n(t)-f_n(t_0)| + |f_n(t_0)-f(t_0)| \\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

$\Rightarrow |f(t)-f(t_0)| < \varepsilon$

$\Rightarrow f$ is continuous

$\Rightarrow f \in C([a,b])$ ∎

The following fact is useful:

Theorem

Let (X, d_X) and (Y, d_Y) be metric spaces and let $\{f_n\}_{n\geq 1}$ be a function sequence defined on X with values in Y

Define $f: X \to Y$

Suppose $f_n \to f$ uniformly over X and that each f_n is continuous over f

Then f is continuous, i.e.

f_n continuous and $f_n \to f$ uniformly $\Rightarrow f$ is continuous